Eco 387L (24): Mathematical Economics Fall 2006
Keys to Homework 1

Exercise 1. We can prove (i) and (ii) based on the definition of sequence convergence in $\mathbb{R}^n$. However, it is a little bit easier to base the proofs on coordinatewise convergence which is equivalent to the original convergence notion.

Lemma 1. If $\sum_{i=1}^n x_i \to 0$ and $x_i \in \mathbb{R}_+ \ \forall i = \overline{1,n}$, then $x_i \to 0 \ \forall i = \overline{1,n}$.
Proof. Here is a proof by contradiction. Let $S^k = \sum_{i=1}^n x_i^k$ be the sequence. If the conclusion is not true, $\exists (i, \varepsilon) : x_i \geq \varepsilon > 0$ in the sense $\neg(\exists k \in \mathbb{N} : x_i^k < \varepsilon)$, and hence $S^k \geq \varepsilon \ \forall k \in \mathbb{N}$. This means S^k is not convergent–a contradiction.

(i) There are two parts. Part 1 ($\Rightarrow$): if $x^k \in \mathbb{R}^n$ coordinatewise converges to x, then $\sum_{i=1}^n \left|x_i^k - x_i\right| \to 0$; Coordinatewise convergence means that $\forall i = \overline{1,n}, x_i^k \to x_i$, hence $\left|x_i^k - x_i\right| \to 0$; by checking with the definition of convergence in $\mathbb{R}$ (details are omitted), $\sum_{i=1}^n \left|x_i^k - x_i\right| \to 0$. Part 2 ($\Leftarrow$): if $\sum_{i=1}^n \left|x_i^k - x_i\right| \to 0$, then $\forall i = \overline{1,n}, x_i^k \to x_i$; by Lemma 1, this is true.

(ii) There are also two parts. Part 1 ($\Rightarrow$): if $x^k \in \mathbb{R}^n$ coordinatewise converges to x, then $\max_{i=\overline{1,n}} \left|x_i^k - x_i\right| \to 0$; we know $\forall i = \overline{1,n}, x_i^k \to x_i$, and the conclusion must be true. Part 2 ($\Leftarrow$): if $\max_{i=\overline{1,n}} \left|x_i^k - x_i\right| \to 0$, then $\forall i = \overline{1,n}, x_i^k \to x_i$; as $\max_{i=\overline{1,n}} \left|x_i^k - x_i\right| \to 0$, $\forall i \ \left|x_i^k - x_i\right| \to 0$, and hence $\forall i$ $x_i^k \to x_i$, which is coordinatewise convergence.

Exercise 2. We check for contraction mapping by its definition. We want to show (WTS): for $x, y \in \mathbb{R}^n$, $d^*(f(x), f(y)) \leq \rho d^*(x, y)$ for $\rho \in [0, 1)$.

Lemma 2. Let $X = \{x_1, .., x_n\} \subset \mathbb{R}^n$ and $Y = \{y_1, .., y_n\} \subset \mathbb{R}^n$, we have $|\max_i\{x_i\} - \max_j\{y_j\}| \leq \max_k\{|x_k - y_k|\}$.
Proof. The result looks intuitive. However, it requires some labor to check its truthfulness. Let i^* and j^* s.t. $x_{i^*} = \max_i\{x_i\}$ and $y_{j^*} = \max_j\{y_j\}$. Without loss of generality, assume $x_{i^*} \geq y_{j^*}$, and $0 \leq |\max_i\{x_i\} - \max_j\{y_j\}| = x_{i^*} - y_{j^*} \leq x_{i^*} - y_{i^*} \leq \max_k\{x_k - y_k\} \leq \max_k\{|x_k - y_k|\}$.

With Lemma 2, we have a direct proof

$$d^*(f(x), f(y)) = \max_{i \in N}\{|f_i(x) - f_i(y)|\}$$

$$= \max_{i \in N} \left\{ \left| \max_{j \in N} \{u(i,j) + \rho x_j\} - \max_{k \in N} \{u(i,k) + \rho y_k\} \right| \right\}$$

$$\leq \max_{i \in N} \left\{ \max_{j \in N} \{|\{u(i,j) + \rho x_j\} - \{u(i,j) + \rho y_j\}|\} \right\}$$

$$= \max_{i \in N} \left\{ \max_{j \in N} \{|\rho x_j - \rho y_j|\} \right\} = \rho \max_{j \in N} \{|x_j - y_j|\} = \rho d^*(x, y).$$

Exercise 3. We need to prove that the Walrasian budget set $B(p, I) = \{x \in \mathbb{R}^n_+ : I \in \mathbb{R}_{++}, p \in \mathbb{R}^n_{++}, p \cdot x \leq I, \}$ is compact. With a continuous utility function $u : B(p, I) \to \mathbb{R}$, compactness of $B(p, I)$ guarantees that the utility maximization problem is valid, i.e. the set of maximizers is not empty.

As compactness implies boundedness, $B(p, I)$ is not compact if $(p \in \mathbb{R}^n_+) \wedge (p \notin \mathbb{R}^n_{++})$ (a contrapositive argument). To see why this is true, note that $\exists i : p_i = 0$; we can buy as much of x_i as we please and still stay in the budget set; this violates boundedness. You can give some illustrative 2-D figure.

To prove compactness, it is sufficient to show closedness and boundedness. To establish boundedness, note that as $\sum_{i=1}^n p_i x_i \leq I$, $\forall i\, p_i x_i \leq I$, i.e. $x_i \leq I/p_i$ for $p_i > 0$; let $M = \sqrt{n} \max_i \{I/p_i + 1 : i = \overline{1, n}\}$. Thus $\forall x \in B(p, I),\ x \in B(0, M)$, i.e. $B(p, I) \subset B(0, M)$, which is boundedness. To have closedness, we rely on the proposition: $S \subset \mathbb{R}^n$ is closed $\Leftrightarrow \forall \{x^k\} : (\forall k\ x^k \in S) \wedge (x^k \to x),\ x \in S$. Pick any convergent sequence $\{x^k\}$ in $B(p, I)$ where $x^k \to x$, we want to show $x \in B(p, I)$. We prove closedness by contradiction. If $x \notin B(p, I)$, $(\exists i : x_i < 0) \vee (\forall i\ x_i \geq 0$ and $p \cdot x = I' > I)$. In either case, $\exists k \in \mathbb{N} : (x^k \in B(p, I)) \wedge (x^k \notin B(p, I))$, which is a contradiction. Here are more details. In the first case, $\exists i : x_i < 0$; by coordinatewise convergence, $\exists k \in \mathbb{N}$ s.t. $x^k \in B(p, I)$ and $|x^k_i - x_i| < |x_i|$, which means $x^k_i < 0$ and $x^k \notin B(p, I)$. In the second case, $\forall i\ x_i \geq 0$ and $p \cdot x = I' > I$; consider an open ball $B(x, r)$ for $r > 0$; $\exists r \in (0, (I' - I)/(n \max(p_i))$ s.t. $\forall x' \in B(x, r), I < p \cdot x'$; as $\{x^k\}$ is a convergent sequence, $\exists k : x^k \in B(x, r)$; thus $x^k \in B(p, I)$ and $p \cdot x^k > I$–a contradiction. You should check why $r < (I' - I)/(n \max(p_i))$ works (hint: $|(x' - x) \cdot p| < I' - I$).

Exercise 4. (Sundaram 1.7.6) WTS: $\lim_{k\to\infty} d(x_k, y_k) = d(x, y)$ for $x_k \to x$ and $y_k \to y$ in $\mathbb{R}^n$. By triangular inequality

$$d(x_k, y_k) \leq d(x_k, x) + d(x, y) + d(y, y_k)$$

$$\Leftrightarrow d(x_k, y_k) - d(x, y) \leq d(x_k, x) + d(y_k, y).$$

Again, by triangular inequality

$$d(x, y) \leq d(x, x_k) + d(x_k, y_k) + d(y_k, y)$$

$$\Leftrightarrow d(x_k, y_k) - d(x, y) \geq -(d(x_k, x) + d(y_k, y)).$$

As $(d(x_k, x) + d(y_k, y)) \to 0$, the inequalities hold in the limit

$$\lim_{k\to\infty} (d(x_k, y_k) - d(x, y)) \leq 0$$

$$\lim_{k\to\infty} (d(x_k, y_k) - d(x, y)) \geq 0.$$

Thus $\lim_{k\to\infty}(d(x_k, y_k) - d(x, y)) = 0$, i.e. $\lim_{k\to\infty} d(x_k, y_k) = d(x, y)$.

Exercise 5. (Sundaram 1.7.29) Part (a) is not necessarily true, while part (b) must be true.

(a) To disprove some statement, all you need to do is giving a counter-example. WTS: if $A \subset \mathbb{R}^2$ is closed, there is a case where $B \subset \mathbb{R}$ is not closed. Graphically, B is the (vertical) shadow of A; thus in principle, a closed set with vertical asymptote(s) is a good candidate because its shadow cannot be closed (but need not be open). One example is the upper contour set of $u(x, y) = xy$ for $u = 1$: $A = \{(x, y) \in \mathbb{R}_{++} \times \mathbb{R}_{++} : y \geq 1/x\}$; by construction $B = (0, \infty)$, which is open. If $A = \{(x, y) \in \mathbb{R}_{++} \times \mathbb{R}_{++} : y \geq 1/x, x \leq 5\}$, $B = (0, 5]$, which is not closed, not open.

(b) WTS: if $A \subset \mathbb{R}^2$ is open, $B \subset \mathbb{R}$ must be open. We have a direct proof by the definition of an open set, i.e. $\forall x \in B$, $\exists r > 0 : B(x, r) \subset B$. The sequence goes as follows. Pick any $x \in B$, by the construction of B, $\exists y \in \mathbb{R} : (x, y) \in A$; as A is open, $\exists s > 0 : B((x, y), s) \subset A$; choose some $s' : 0 < s' < s$ and construct the closed ball $\overline{B}((x, y), s')$; by the construction of B, the projection $\mathrm{Pr}_x \, \overline{B}((x, y), s')$ of $\overline{B}((x, y), s')$ on the x-axis includes x and is a subset of B; choose $r : 0 < r < s'$ and construct the subset $B(x, r) \subset \mathrm{Pr}_x \, \overline{B}((x, y), s') \subset B$. Note that we are working on the 2-D Euclidean space, and the projection of a closed ball on the x-axis is what we already know; thus the details are omitted here. The proof relies heavily on construction, which is not necessarily unique.