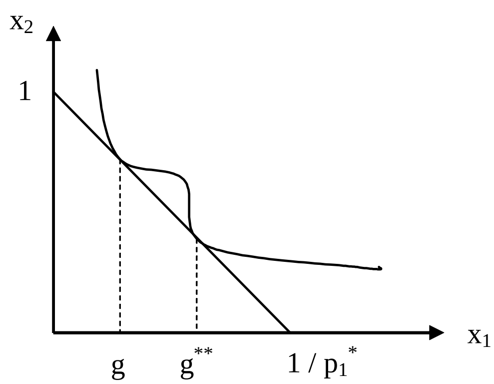
Eco 387L (24): Mathematical Economics Fall 2006
Keys to Homework 2

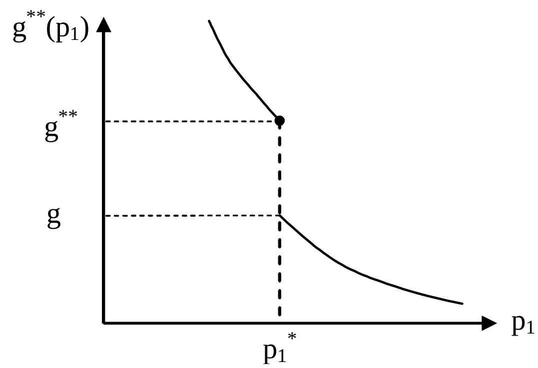
Exercise 1. (You can have your own examples) Let's look at a utility maximization problem where there are: two commodities with strictly positive prices $p = (p_1, p_2)$, a strictly positive budget $I = 1$, and nonconvex indifference curves (Figure 1). In this example, commodity 2 is the numeraire, i.e. $p_2 = 1$. The parameter space is $\Theta = \{p_1 \in \mathbb{R}_{++}\}$. The commodity space is $S = \mathbb{R}^2_+$. The budget set for each p_1 is $D(p_1) = \{x \in \mathbb{R}^2_+ : p \cdot x \leq I\}$. Let $g(x, \theta) = x_1$ (x-coordinate of point x). Note the following: (i) we can define a utility function $f : S \times \Theta \to \mathbb{R}$ which generates those indifference curves and only depends continuously on points (commodity bundles) in S; (ii) $g : S \times \Theta \to \mathbb{R}$ is continuous; (iii) $D : \Theta \to 2^S$ is a compact-valued continuous correspondence.

Figure 1



Starting from p_1^*, move p_1 in both directions, we can construct $g^{**}(p_1)$ and $D^{**}(p_1)$. Figure 2 shows $g^{**}(p_1)$, which is discontinuous at p_1^*. Note that $g^{**}(p_1)$ also shows the x-coordinates of the points in $D^{**}(p_1)$. By the original definition or the sequence argument of uhc, $D^{**}(p_1)$ is not uhc at p_1^*.

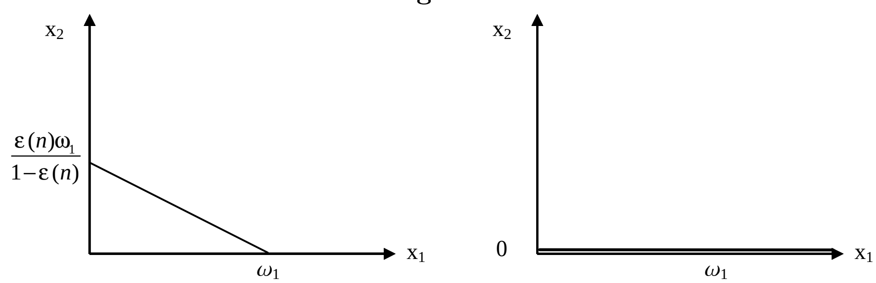
Figure 2



Exercise 2. Again, let's look at a 2-commodity example. As $\omega \in \mathbb{R}^2_+$ and $\omega \notin \mathbb{R}^2_{++}$, we have 2 cases, one, wlog, $\omega = (\omega_1, 0)$, the other $\omega = \mathbf{0}$ (vector zero). We will check for lhc at $p = (0, 1)$. Construct the sequence $p^{\varepsilon(n)} = (\varepsilon(n), 1 - \varepsilon(n))$. Note that as $\varepsilon(n) \to 0$, $p^{\varepsilon(n)} \to (0, 1)$. We will go over the cases one by one (the division into cases is just for clarification).

Figure 3 shows Case 1 for $\varepsilon(n) > 0$ (Panel A) and for $\varepsilon(\infty) = 0$ (Panel B). We see that for $\varepsilon(n) > 0$, the budget set is bounded; for $\varepsilon(\infty) = 0$, the budget set "explodes" in the x_1-dimension. Pick an open set s.t. $x_1 \in (\omega_1 + 1, \omega_1 + 2)$ and $x_2 = 0$, i.e. a set V corresponding to $p = (0, 1)$. No matter how large $\varepsilon(n)$ is, $B(p^{\varepsilon(n)}, \omega) \cap V = \emptyset$, i.e. the budget correspondence is not lhc at $p = (0, 1)$.

Figure 3



In Case 2, for $p >> 0$, the budget set is $(0, 0)$. For $p = (0, 1)$, again, the budget set blows up. It's now even easier to show the budget correspondence is not lhc at $p = (0, 1)$.

Exercise 3. We are looking at the behavior of player i in a normal form game. For player i, the parameter space is $\Theta_i = \sum_{-i} = \times_{j \neq i} \sum_j$, and $\theta_i = \sigma_{-i} = \times_{j \neq i} \sigma_j$; the choice set is $\sum_i = \{\sigma_i : S_i \to [0, 1] : \sum_{s_i \in S_i} \sigma_i(s_i) = 1\}$, which does not depend on Θ_i (choices of other players); the utility function f_i is $\widehat{r}_i : \sum_i \times \Theta_i \to \mathbb{R}$ as $\widehat{r}_i(\sigma) = \sum_{s \in S} [\sigma_1(s_1)..\sigma_n(s_n)]\, r_i(s)$.

(i) WTS: $\widehat{r}_i(\sigma)$ is a continuous function at all $\sigma \in \sum = \times_{i \in N} \sum_i$. By construction, $\sum$ is a compact convex set, and $\widehat{r}_i(\sigma)$ is a linear function in σ_j $\forall j \in N$. Thus $\widehat{r}_i(\sigma)$ is continuous.

(ii) WTS: $D : \Theta_i \to P(\sum_i)$ is a convex-valued compact-valued continuous correspondence. Note that $D(\theta_i) = \sum_i$ $\forall \theta_i \in \Theta_i$. By construction, $\sum_i$ is both compact-valued and convex-valued. As D is a constant correspondence, D is continuous (easy to check for both lhc and uhc).

(iii) WTS: $\widehat{r}_i(\sigma_i, \sigma_{-i})$ is a concave function in σ_i. Note that a linear function is readily concave, and not strictly concave.

(iv) WTS: $D : \Theta_i \to P(\sum_i)$ has a convex graph. Note that Θ_i and $\sum_i$ are convex. Pick $(\theta_i^1, \sigma_i^1), (\theta_i^2, \sigma_i^2) \in Gr(D)$. It is easy to show that $\lambda(\theta_i^1, \sigma_i^1) + (1 - \lambda)(\theta_i^2, \sigma_i^2) \in Gr(D)$ $\forall \lambda \in (0, 1)$.

Exercise 4. Checking the conditions of the Maximum Theorem.

(a) Not all conditions are met.

(i) WTS: f is not continuous at $(x, \theta) = (0, 0)$. Note that $f(0, 0) = 0$. Pick equivalent sequences θ^n and x^n s.t. $\theta^n \to 0$ and $x^n = \theta^n \ \forall n \in \mathbb{N}$. Observe that $f(x^n, \theta^n) = 1 \ \forall x^n = \theta^n > 0$. Thus f is not continuous.

(ii) WTS: D is not lhc at $\theta = 1/2$. We prove this by contradiction. Suppose D is lhc at $\theta = 1/2$; pick $s = 1/2 \in D(\theta)$; consider the sequence $\theta^n \to \theta$ s.t. $\theta^n \leq \theta \ \forall n \in \mathbb{N}$; observe that $\nexists s^n : (s^n \in D(\theta^n) \wedge \ s^n \to s)$, a contradiction. Thus, lhc is violated, and D is not a continuous correspondence.

(b) Observe that $D^*(\theta) \neq \emptyset \ \forall \theta \in \Theta$. Specifically $D^*(0) = [0, 1]$; $D^*(\theta) = \theta$ for $\theta \in (0, \theta_1]$; $D^*(\theta) = 1 - 2/\theta$ for $\theta \in (\theta_1, 0.5)$; $D^*(\theta) = \theta$ for $\theta \in [0.5, \theta_2]$; $D^*(\theta) = 2 - 2\theta$ for $\theta \in (\theta_2, 1]$. Next observe that D^* is neither uhc nor lhc (Figure 4).

Figure 4