Eco 387L (24): Mathematical Economics Fall 2006
Keys to Homework 3

Exercise 1. We check for definiteness by Proposition 8.1 (lecture note). There are three cases:

(a) $f(x) = x_1^2 + x_2^{1/2}$

$$D^2 f(x) = \begin{bmatrix} 2 & 0 \\ 0 & -0.25x^{-3/2} \end{bmatrix},$$

$$D^2 f(1,1) = \begin{bmatrix} 2 & 0 \\ 0 & -0.25 \end{bmatrix}.$$

This matrix is not positive definite, not negative definite , not positive semidefinite, not negative semidefinite. It is indefinite.

(b) $f(x) = x_1^{1/2} x_2^{1/2}$

$$D^2 f(x) = 0.25 \begin{bmatrix} -x_1^{-3/2} x_2^{1/2} & x_1^{-1/2} x_2^{-1/2} \\ x_1^{-1/2} x_2^{-1/2} & -x_1^{1/2} x_2^{-3/2} \end{bmatrix}, \ x \in \mathbb{R}^2_{++}.$$

This matrix is not positive definite, not negative definite, not positive semidefinite. It is negative semidefinite.

(c) $f(x) = x_1^2 x_2^2$

$$D^2 f(x) = 2 \begin{bmatrix} x_2^2 & 2x_1x_2 \\ 2x_1x_2 & x_1^2 \end{bmatrix}, \ x \in \mathbb{R}^2_{++}.$$

This matrix is not positive definite, not negative definite, not positive semidefinite, not negative semidefinite. It is indefinite.

Exercise 2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ as $f(x,y) = ax^2 + by^2 + 2cxy + d$. We need conditions for f to be concave. From Proposition 10.2 (lecture note), the function $f(x)$ function is concave iff $D^2 f(x)$ is negative semidefinite everywhere in the domain. We have the Hessian

$$D^2 f(x) = 2 \begin{bmatrix} a & c \\ c & b \end{bmatrix}.$$

First, observe that the Hessian does not depend on d. Intuitively, d is the shifting factor and should not affect the convexity of the function. To have $D^2 f(x)$ to be negative semidefinite, we must have $a \leq 0$, $b \leq 0$, and $ab - c^2 \geq 0$.

Exercise 3. We are looking at a case where the utility function has strictly concave indifference curves so that for each price vector $p \in \mathbb{R}^{n+1}_{++}$ the max

$x(p, w)$ is a singleton. Given some $p = (p_1, .., p_n)$, the marginal rate of substitution between any two goods at $x(p, w)$, e.g. u_i/u_j, does not change when we vary w (wealth or income) (why?). The Engel curve for a given p connects those $x(p, w)$ for different w. Thus the Engel curve gets another name: the income expansion path. The vector $\psi \in \mathbb{R}^n_{++}$ can be understood as

$$\psi = \begin{bmatrix} p_1/p_0 \\ .. \\ p_n/p_0 \end{bmatrix}.$$

In this exercise the Engel curve for some p is a line (not necessarily linear) in $\mathbb{R}^{n+1}_{++}$. That means given (x_0, ψ), we can uniquely pin down the corresponding Engel curve and all the other coordinates $(x_1, .., x_n)$. Given some point x, which is defined by (x_0, ψ), we want to express $x_1, .., x_n$ as functions of (x_0, ψ) (in the neighborhood). Define $F : \mathbb{R}^{1+n+n}_{++} \to \mathbb{R}^n_{++}$ as $F(x_0, \psi', \widetilde{x}) = \varphi(x) - \psi = \mathbf{0}$ where $\widetilde{x} = (x_1, .., x_n)$ and ψ' is a row. Note that

$$F_{x_0} = \begin{bmatrix} (u_{10}u_0 - u_1u_{00}) / (u_0)^2 \\ .. \\ (u_{n0}u_0 - u_nu_{00}) / (u_0)^2 \end{bmatrix}_{n\times 1},$$

$$F_{\psi} = \begin{bmatrix} -1 & .. & 0 \\ .. & .. & .. \\ 0 & .. & -1 \end{bmatrix}_{n\times n} = -1 \times I(n),$$

$$F_{\widetilde{x}} = \begin{bmatrix} (u_{11}u_0 - u_1u_{01}) / (u_0)^2 & .. & (u_{1n}u_0 - u_1u_{0n}) / (u_0)^2 \\ .. & .. & .. \\ (u_{n1}u_0 - u_nu_{01}) / (u_0)^2 & .. & (u_{nn}u_0 - u_nu_{0n}) / (u_0)^2 \end{bmatrix}_{n\times n}.$$

(i) We need the condition to apply the implicit function theorem. It is $Det(F_{\widetilde{x}}) \neq 0$ or $F_{\widetilde{x}}$ is invertible.

(ii) Applying the implicit function theorem, there exists $g : \mathbb{R}^{1+n}_{++} \to \mathbb{R}^n_{++}$ such that $\widetilde{x} = g(x_0, \psi')$. In addition $Dg(x_0, \psi') = (F_{\widetilde{x}})^{-1}[F_{x_0} \quad F_{\psi}]$ (dimension: $n \times (1 + n)$). Finally, the slope vector of the Engel curve given (x_0, ψ) is $D_{x_0}g(x_0, \psi') = (F_{\widetilde{x}})^{-1}F_{x_0}$ (dimension: $n \times 1$).

Note: I write $(1 + n + n)$ and $(1 + n)$ to emphasize where they come from.