Eco 387L (24): Mathematical Economics Fall 2006
Keys to Homework 4

Exercise 1. (Sundaram 5.8.6, p142) Equality-constrained optimization

a) Let $\lambda_{m\times 1}$ be the Lagrange multiplier vector. The Lagrangean is

$$L(x,\lambda) = c'x + \frac{1}{2}x'Dx + \lambda'(b - Ax).$$

The first-order conditions are

$$\begin{aligned} & DL(x^*,\lambda^*) = 0 \\ \iff & \begin{cases} D_x L(x^*,\lambda^*) = 0_{n\times 1} \\ D_\lambda L(x^*,\lambda^*) = 0_{m\times 1} \end{cases} \\ \iff & \begin{cases} c + Dx^* - A'\lambda^* = 0_{n\times 1} \\ b - Ax^* = 0_{m\times 1} \end{cases} . \end{aligned}$$

b) From the FOC, we now solve for x^*. Actually, we need λ^* first. Note that as D is symmetric and ND, we know D^{-1} exists. Thus

$$c + Dx^* - A'\lambda^* = 0_{n\times 1} \quad (1)$$

$$\iff D^{-1}c + x^* - D^{-1}A'\lambda^* = 0_{n\times 1}$$

$$\iff AD^{-1}c + Ax^* - AD^{-1}A'\lambda^* = 0_{n\times 1}$$

$$\iff \left(AD^{-1}A'\right)\lambda^* = AD^{-1}c + b$$

$$\implies \lambda^* = \left(AD^{-1}A'\right)^{-1}\left(AD^{-1}c + b\right), \quad (2)$$

where we assume $\left(AD^{-1}A'\right)^{-1}$ exists. Substitute λ^* from (2) into (1)

$$c + Dx^* = A'\left(AD^{-1}A'\right)^{-1}\left(AD^{-1}c + b\right)$$

$$\iff Dx^* = A'\left(AD^{-1}A'\right)^{-1}\left(AD^{-1}c + b\right) - c$$

$$\iff x^* = D^{-1}\left[A'\left(AD^{-1}A'\right)^{-1}\left(AD^{-1}c + b\right) - c\right]. \quad (3)$$

Exercise 2. (Sundaram 6.6.4, p169) Inequality-constrained optimization

Let λ be the Kuhn-Tucker multiplier for $\sum_{t=1}^{T} x_t \leq 1$; μ_t be the K-T multiplier for x_t, $t = 1, ..., T$. Notation: $\mu = \{\mu_1, ..., \mu_T\}$. The Lagrangean is

$$L(x,\lambda,\mu) = \sum_{t=1}^{T}\left(\frac{1}{2}\right)^t x_t^{1/2} + \lambda\left(1 - \sum_{t=1}^{T} x_t\right) + \sum_{t=1}^{T}\mu_t x_t.$$

Next, we find the critical points of function L. The critical points satisfy the following first-order conditions

$$\frac{\partial L}{\partial x_t}(x,\lambda,\mu) = \left(\frac{1}{2}\right)^{t+1} x_t^{-1/2} - \lambda + \mu_t = 0,\ t = 1,...,T \tag{4}$$

$$\frac{\partial L}{\partial \lambda}(x,\lambda,\mu) = 1 - \sum_{t=1}^{T} x_t \geq 0,\ \lambda \geq 0,\ \lambda\left(1 - \sum_{t=1}^{T} x_t\right) = 0 \tag{5}$$

$$\frac{\partial L}{\partial \mu_t}(x,\lambda,\mu) = x_t \geq 0,\ \mu_t \geq 0,\ \mu_t x_t = 0,\ t = 1,...,T. \tag{6}$$

Observe the following $\sqrt{x_t}$ is strictly increasing and strictly concave function where the slope at 0 is infinite. First, WTS: the maximization requires $\sum_{t=1}^{T} x_t = 1$. Suppose not, i.e. $\sum_{t=1}^{T} x_t < 1$, we can always improve the value by increasing some x_t within the constraint. Thus the equality must hold and we infer that not all $x_t = 0$. Second, WTS: the maximization solution requires that $x_t > 0\ \forall t$. Suppose some $x_t = 0$. Again, we can improve the value by increasing x_t and decreasing some x_k infinitestimally by the same amount, where $x_k > 0$ (the slope at $x_k > 0$ is smaller than the slope at $x_t = 0$). From (6), $\mu_t = 0\ \forall t$. From (4), $\lambda > 0$ and

$$\frac{x_1}{0.5^{2(1+1)}} = ... = \frac{x_t}{0.5^{2(t+1)}} = ... = \frac{x_T}{0.5^{2(T+1)}} = \lambda^{-2} = \frac{\sum_{t=1}^{T} x_t}{\sum_{t=1}^{T} 0.25^{(t+1)}},$$

where the last equality comes from $a/b = c/d = (a+c)/(b+d)$. Finally

$$x_t = \frac{0.25^{(t+1)}}{\sum_{t=1}^{T} 0.25^{(t+1)}},\ t = 1,...,T.$$

Exercise 3. (Sundaram 7.8.19, p200) Role of convexity in optimization

Let λ_t be the K-T multiplier for the resource constraint at t, μ_t for c_t, and γ_t for x_t, $t = 1,...,T$. Notation: $\lambda = \{\lambda_1,...,\lambda_T\}$, $\mu = \{\mu_1,...,\mu_T\}$, and $\gamma = \{\gamma_1,...,\gamma_T\}$. We have the Lagrangean

$$\begin{aligned} L(c,x,\lambda,\mu,\gamma) &= \sum_{t=1}^{T} u(c_t) + \lambda_1(x - c_1 - x_1) + \sum_{t=2}^{T} \lambda_t (f(x_{t-1}) - c_t - x_t) \\ &\quad + \sum_{t=1}^{T} \mu_t c_t + \sum_{t=1}^{T} \gamma_t x_t. \end{aligned}$$

The K-T FOC is

$$L_{c_1} = u'(c_1) - \lambda_1 + \mu_1 = 0$$
$$L_{c_t} = u'(c_t) - \lambda_t + \mu_t = 0,\ t = \overline{2,T}$$

$$L_{x_1} = -\lambda_1 + \lambda_2 f'(x_1) + \gamma_1 = 0$$
$$L_{x_t} = -\lambda_t + \lambda_{t+1} f'(x_t) + \gamma_t = 0,\ t = \overline{2, T-1}$$
$$L_{x_T} = -\lambda_T + \gamma_T = 0$$
$$L_{\lambda_1} = x - c_1 - x_1 \geq 0,\ \lambda_1 \geq 0,\ \lambda_1 L_{\lambda_1} = 0$$
$$L_{\lambda_t} = f(x_{t-1}) - c_t - x_t \geq 0,\ \lambda_t \geq 0,\ \lambda_t L_{\lambda_t} = 0,\ t = \overline{2, T}$$
$$L_{\mu_t} = c_t \geq 0,\ \mu_t \geq 0,\ \mu_t c_t = 0,\ t = \overline{1, T}$$
$$L_{\gamma_t} = x_t \geq 0,\ \gamma_t \geq 0,\ \gamma_t x_t = 0,\ t = \overline{1, T}.$$

We need to impose further conditions so that the K-T conditions are sufficient for a max. Define $U \subset \mathbb{R}_+^T \times \mathbb{R}_+^T$ with a point in U is some (c, x). Define $g : U \to \mathbb{R}^+$ where $g(c, x) = \sum_{t=1}^T u(c_t)$. We already have that u and f are nondecreasing and continuous functions from $\mathbb{R}^+$ to $\mathbb{R}^+$. To use the Theorem of Kuhn-Tucker under Convexity (Sundaram p187), the regularity conditions needed are: (i) g is a concave C^1 function, this will be satisfied if u is an increasing concave C^1 function; (ii) f is an increasing concave C^1 function; and (iii) the set U is open and convex, which means $c_t > 0$ and $x_t > 0$ $\forall t$.