Eco 387L (24): Mathematical Economics Fall 2006
Keys to Midterm 1

The total number of points is 110. Here are the quick answers:

Question 1. (50 points) Checking different basic concepts

Part (i) (10 points) The negation is

$$\exists x \in X, \forall y \in Y, \exists z \in Z; f(x,y) \geq g(y,z) \wedge h(x,y,z) < 0.$$

Part (ii) (10 points) It is not difficult to see that $\sum_{t=1}^{T'} y_t > \sum_{t=1}^{T'} x_t \ \forall T' > (2.001 \times 10^{13} + 1)$, i.e. $\{y_t\} \succsim \{x_t\}$. This also means that $\{x_t\} \succsim \{y_t\}$ is not true. In combination, $\{y_t\} \succ \{x_t\}$.

Part (iii) (10 points) It is straightforward to prove by contradiction. Suppose the limit $x < 0$. As the sequence is converging to x, we can find some x_n close enough to x s.t. $x_n < 0$. This contradicts the condition that $x_n \geq 0 \ \forall n$. Thus $x \geq 0$.

Part (iv) (10 points) We prove continuity by definition with the $\epsilon - \delta$ argument. Function $f + g : \mathbb{R} \to \mathbb{R}$ is continuous if it is continuous at every point in $\mathbb{R}$. Pick an arbitrary point $y \in \mathbb{R}$. WTS: $\forall \epsilon > 0, \exists \delta > 0; d(x,y) < \delta \to d((f+g)(x),(f+g)(y)) < \epsilon$. Notation: $(f+g)(x) = f(x) + g(x)$. As f and g are continuous, we can pick some δ s.t. $d(x,y) < \delta \to d(f(x), f(y)) < \epsilon/2$ and $d(g(x), g(y)) < \epsilon/2$.

$$\begin{aligned}
& d((f+g)(x),(f+g)(y)) \\
= \ & |(f+g)(x) - (f+g)(y)| \\
= \ & |f(x) + g(x) - f(y) - g(y)| \\
= \ & |(f(x) - f(y)) + (g(x) - g(y))| \\
\leq \ & |(f(x) - f(y))| + |(g(x) - g(y))| \ \text{(Triangle Inequality)} \\
= \ & d(f(x), f(y)) + d(g(x), g(y)) \\
< \ & \epsilon.
\end{aligned}$$

Part (v) (10 points) We need to prove $g \circ f : \mathbb{R}^n_+ \to \mathbb{R}$ is strictly concave. Pick any $x, y \in \mathbb{R}^n_+$ s.t. $x \neq y$ and some $\lambda \in (0,1)$. WTS: $(g \circ f)(\lambda x + (1-\lambda)y) > \lambda (g \circ f)(x) + (1-\lambda)(g \circ f)(y)$.

$$\begin{aligned}
& (g \circ f)(\lambda x + (1-\lambda)y) \\
= \ & g(f(\lambda x + (1-\lambda)y)) \\
> \ & g(\lambda f(x) + (1-\lambda) f(y)) \ \text{(strictly concave } f\text{, strictly increasing } g\text{)} \\
\geq \ & \lambda g(f(x)) + (1-\lambda) g(f(y)) = \lambda (g \circ f)(x) + (1-\lambda)(g \circ f)(y) \ \text{(concave } g\text{)}.
\end{aligned}$$

Question 2. (20 points, 6 for best response, 2 for NE, 2 for closedness) To complete the answer, it is better to construct the best-response functions, first for $\theta \in (-1, 1]$ and then $\theta = -1$. Note that the payoff structure is symmetric. WLOG, the first player is the column player (we need this for payoff specification). Let $p \in [0, 1]$ be the probability that the column player plays L. Let $q \in [0, 1]$ be the probability that the row player plays T. It is easier to solve for NE by the 2-D graph. Notation: $\Phi(\theta)$ is the set of NE for θ and $\sigma = (p, q)$ constitutes a strategy profile. For $\theta \in (-1, 1]$, $\Phi(\theta) = \{(0, 0), (1, 1), (\frac{1+\theta}{2+\theta}, \frac{1}{2+\theta})\}$. For $\theta = -1$, $\Phi(-1) = \{(p = 0, q \in [0, 1]), (p \in [0, 1], q = 1)\}$, i.e. there is a continuum of NE. Actually, as $\theta \to -1$, the set of NE blows up from a 3-point set to a continuum (upper hemicontinuous at $\theta = -1$). Notation: $Gr(\Phi) = \{(\theta, \sigma) \in [-1, 1] \times [0, 1]^2 : \sigma \in \Phi(\theta)\}$ is the graph. It is easy to see in a 3-D graph that $Gr(\Phi)$ is closed.

Question 3. (30 points) Checking for some function properties

Part (i) (10 points) Pick any $x, y \in \mathbb{R}^n$ s.t. $x \neq y$ and some $\lambda \in (0, 1)$. WTS: $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$ and the equality occurs for some (x, y). Note that $\arg\max_p x \cdot p$ is not empty and not necessarily unique. Let $p_{xy} = \arg\max_p(\lambda x + (1-\lambda)y) \cdot p$, $p_x = \arg\max_p x \cdot p$, and $p_y = \arg\max_p y \cdot p$. So

$$\begin{aligned} & f(\lambda x + (1-\lambda)y) \\ = \ & (\lambda x + (1-\lambda)y) \cdot p_{xy} \\ = \ & \lambda x \cdot p_{xy} + (1-\lambda)y \cdot p_{xy} \\ \geq \ & \lambda x \cdot p_x + (1-\lambda)y \cdot p_y \\ = \ & \lambda f(x) + (1-\lambda)f(y). \end{aligned}$$

The equality happens for many cases. You can check for $x = (1, .., 1)$ and $y = \alpha x$ where the scalar $\alpha > 1$.

Part (ii) (10 points) We can apply the Maximum Theorem treating x as the parameter and p as the choice variable. We only need to verify the conditions of the Theorem.

Part (iii) (10 points) We prove by contradiction. Suppose that $P \neq Q$. WLOG, $\exists p \in P \backslash Q$. Applying the Separating Hyperplane Theorem, $\exists x \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $\exists \alpha \in \mathbb{R}$ s.t. $\forall q \in Q$ we have $x \cdot q \geq \alpha > x \cdot p$. This means $\min_{p \in P} x \cdot p < \min_{q \in Q} x \cdot q$, a contradiction to the hypothesis $\min_{p \in P} x \cdot p = \min_{q \in Q} x \cdot q$. Thus $P = Q$.

Question 4. (20 points) The question is for proving the existence of a maximal point by a sequence of arguments. Suppose there is no maximal point, i.e. $P(x) \cap X \neq \emptyset \ \forall x \in X$. Next, our task is to fill in two holes in the sequence.

Part (i) (10 points) WTS: $\forall x \in X$, $\exists y_x \in X$ and open $U_x \subset \mathbb{R}^n$ with $x \in U_x$ s.t. $y_x \in P(x') \cap X \ \forall x' \in U_x \cap X$. Pick any $x \in X$. As $P(x) \cap X \neq \emptyset$, pick $y_x \in P(x) \cap X$ and hence $x \in P^{-1}(y_x)$. As $P^{-1}(y_x)$ is open, we can construct an open set U_x containing x s.t. $U_x \subset P^{-1}(y_x)$. Note that $\forall x' \in U_x \cap X \subset P^{-1}(y_x)$, $y_x \in P(x')$. In combination, we have the result.

Part (ii) (10 points) WTS: $f(x) \neq x \ \forall x \in X$. Pick any $x \in X$. Construct the index subset $\Omega \subset \{1, 2, .., n\}$ s.t. $x \in U_{x_j} \ \forall j \in \Omega$. Note that $y_{x_j} \in P(x) \ \forall j \in \Omega$ and $\pi_k(x) = 0 \ \forall k \notin \Omega$. Thus

$$\begin{aligned} f(x) &= \sum_{i=1}^{n} \pi_i(x) y_{x_i} \\ &= \sum_{j \in \Omega} \pi_j(x) y_{x_j}. \end{aligned}$$

As $y_{x_j} \in P(x) \ \forall j \in \Omega$ and $P(x)$ is convex, $f(x) \in P(x)$. By irreflexivity, $x \notin P(x)$ which means $f(x) \neq x$.

Note: I use some symbols for grading. Here are they:

$$\begin{aligned} -m &= -1 \\ -h &= -2 \\ -b &= -3 \\ -b' &= -4 \\ -n &= -5 \\ -t' &= -8 \\ -m` &= -10 \end{aligned}$$