**Eco 387L (24): Mathematical Economics Fall 2006**
**Keys to Midterm 2**

The total number of points is 100.

**Question 1.** (30 points) We follow the procedure: (i) setup the Lagrangean; (ii) find the candidates with the FOC; and (iii) pick the candidate(s) that maximize the objective function. The Lagrangean is

$$\begin{aligned} L(x, y, \lambda_1, \lambda_2, \lambda_3) &= 2\ln x + 5\ln y \\ &\quad +\lambda_1 (6 - x - y) \\ &\quad +\lambda_2 (10 - x - 2y) \\ &\quad +\lambda_3 (9 - 2x - y). \end{aligned}$$

The FOC's are

$$L_x = 2/x - \lambda_1 - \lambda_2 - 2\lambda_3 = 0$$
$$L_y = 5/y - \lambda_1 - 2\lambda_2 - \lambda_3 = 0$$
$$L_{\lambda_1} = 6 - x - y \geq 0;\ \lambda_1 \geq 0;\ \lambda_1 (6 - x - y) = 0$$
$$L_{\lambda_2} = 10 - x - 2y \geq 0;\ \lambda_2 \geq 0;\ \lambda_2 (10 - x - 2y) = 0$$
$$L_{\lambda_3} = 9 - 2x - y \geq 0;\ \lambda_3 \geq 0;\ \lambda_3 (9 - 2x - y) = 0.$$

First note that we cannot have 3 inequality constraints bind at the same time (Why?). In addition, at least one of the constraints binds (Why?). Thus, in principle, there are 6 cases to look at: (1) only $\lambda_1 > 0$; (2) only $\lambda_2 > 0$; (3) only $\lambda_3 > 0$; (4) only $\lambda_1, \lambda_2 > 0$; (5) only $\lambda_1, \lambda_3 > 0$; and (6) only $\lambda_2, \lambda_3 > 0$. For each case, it is straightforward to find the candidate(s) $(x_i, y_i)$ and the corresponding objective value denoted by $V_i$ for $i = 1, ..., 6$. Here are the results:

C(1): $x = 12/7;\ y = 30/7;\ \Longrightarrow$ outside the constraint set.
C(2): $x = 20/7;\ y = 25/7;\ \Longrightarrow$ outside the constraint set.
C(3): $x = 9/7;\ y = 45/7;\ \Longrightarrow$ outside the constraint set.
C(4): $x = 2;\ y = 4;\ V_4 = 8.32.$
C(5): $x = 3;\ y = 3;\ V_5 = 7.69.$
C(6): $x = 8/3;\ y = 11/3;\ \Longrightarrow$ outside the constraint set.

Finally, the solution is $(x = 2; y = 4)$; the optimal value is 8.32.

**Question 2.** (30 points) We also follow the "cookbook" procedure. The Lagrangean is

$$L(x, y, \lambda, \mu_1, \mu_2) = (x + 1)(y + 1) + \lambda (I - px - qy) + \mu_1 x + \mu_2 y. \tag{1}$$

The FOC's are

$$L_x = (y + 1) - \lambda p + \mu_1 = 0 \tag{2}$$
$$L_y = (x + 1) - \lambda q + \mu_2 = 0 \tag{3}$$

$$L_\lambda = I - px - qy = 0;\ \lambda \geq 0;\ \lambda\,(I - px - qy) = 0 \tag{4}$$

$$L_{\mu_1} = x \geq 0;\ \mu_1 \geq 0;\ \mu_1 x = 0 \tag{5}$$

$$L_{\mu_2} = y \geq 0;\ \mu_2 \geq 0;\ \mu_2 y = 0. \tag{6}$$

It is straightforward to see that $\lambda > 0$; $px + qy = I$; $x$ and $y$ cannot both be zero. Thus, at the optimal point(s), it is either $(x = 0, y > 0)$, or $(x > 0, y = 0)$, or $(x > 0, y > 0)$. The final solutions depend on the parameters $p, q$, and $I$, which are all strictly positive. There are 3 cases regarding $p$ and $q$: (i) $p = q$; (ii) $p > q$; and (iii) $q > p$.

Case 1: $p = q$. First, consider $(x = 0, y > 0)$. The candidate is $(x = 0, y = I/q)$ and the objective value is $(I/q + 1)$. Second, consider $(x > 0, y > 0)$. The candidate is $(x = I/(2q), y = I/(2q))$ and the objective value is $(I/(2q) + 1)^2$. The latter candidate always yields a strictly higher value than the former. Note that there's no need to consider $(x > 0, y = 0)$ (Why?). Thus, for $p = q$, $(x^* = I/(2q), y^* = I/(2q))$.

Case 2: $p > q$. First, consider $(x = 0, y > 0)$. The candidate is $(x = 0, y = I/q)$ and the objective value is $(I/q + 1)$. Second, consider $(x > 0, y = 0)$. The candidate is $(x = I/p, y = 0)$ and the objective value is $(I/p+1)$. Third, consider $(x > 0, y > 0)$. The candidate is $(x = (I - p + q)/(2p), y = (I + p - q)/(2q))$ and the objective value is $(I + p + q)^2/(4pq)$.

As $p > q$, the second candidate is dominated by the first candidate. That means we only need to compare the first and the third. Note that the third candidate violates $x > 0$ for $I \leq p - q$. Thus for $I \leq p - q$, the only candidate left is the first, i.e. $(x = 0, y = I/q)$. For $I > p - q$, the third candidate dominates the first because

$$\begin{aligned}
&(I + p + q)^2/(4pq) > (I/q + 1) \\
\iff\ & I^2 + p^2 + q^2 + 2Ip + 2Iq + 2pq > 4Ip + 4pq \\
\iff\ & (I^2 - Ip + Iq) - (Ip - p^2 + pq) + (Iq - pq + q^2) > 0 \\
\iff\ & I(I - p + q) - p(I - p + q) + q(I - p + q) > 0 \\
\iff\ & (I - p + q)^2 > 0.
\end{aligned}$$

In combination, the results for $p > q$ are

$$\begin{array}{ll}
\text{if } I \leq p - q: & x^* = 0,\ y^* = I/q \\
\text{if } I > p - q: & x^* = (I - p + q)/(2p),\ y^* = (I + p - q)/(2q).
\end{array}$$

Case 3: $p < q$. By the same token, the results are

$$\begin{array}{ll}
\text{if } I \leq q - p: & x^* = I/p,\ y^* = 0 \\
\text{if } I > q - p: & x^* = (I - p + q)/(2p),\ y^* = (I + p - q)/(2q).
\end{array}$$

**Question 3.** (20 points) Construct the function

$$F(z,x,y)=\left[\begin{array}{c} x^2-y^3+z^4 \\ x+y^2-z^3 \end{array}\right].$$

Note that $F$ is a $C^1$ function defined on the open set $\mathbb{R}^3$. Consider some $(z,x,y)$ s.t. $F(z,x,y)=[1\ 1]'$. To have $x$ and $y$ as functions in the neighborhood $U\subset\mathbb{R}$ of $z$, we need $D_{x,y}F(z,x,y)$ to be invertible. Specifically

$$D_{x,y}F(z,x,y)=\left[\begin{array}{cc} 2x & -3y^2 \\ 1 & 2y \end{array}\right].$$

The condition is $4xy+3y^2\neq 0$, i.e. $y\neq 0$ and $x/y\neq -3/4$. By the Implicit Function Theorem, $x$ and $y$ can be solved for as functions of $z$ in the neighborhood $U$. Let

$$D_zF(z,x,y)=\left[\begin{array}{c} 4z^3 \\ -3z^2 \end{array}\right].$$

Finally, the derivatives are

$$\left[\begin{array}{c} \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial z} \end{array}\right]=-\left[D_{x,y}F(z,x,y)\right]^{-1}D_zF(z,x,y).$$

**Question 4.** (20 points) Consider a $C^2$ function $u:\mathbb{R}^n_{++}\to\mathbb{R}$ s.t.

$$u\left(x\right)=\sum_{i=1}^{n}v_i\left(x_i\right)$$

where $v_i:\mathbb{R}_{++}\to\mathbb{R}$, $i=1,...,n$. WTS: $u$ is concave iff $v_i''\leq 0\ \forall i$. By Theorem 7.10 (Sundaram), $u$ is concave iff $D^2u(x)$ is NSD $\forall x\in\mathbb{R}^n_{++}$. Given some $x\in\mathbb{R}^n_{++}$, the Hessian of $u$ is the diagonal matrix

$$D^2u(x)=\left[\begin{array}{cccc} v_1''(x_1) & 0 & ... & 0 \\ 0 & v_2''(x_2) & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & v_n''(x_n) \end{array}\right]_{n\times n}.$$

In addition, we check for NSD by Theorem 1.63 (Sundaram): $(-1)^k|A_k^\pi|\geq 0\ \forall k,\pi$, where $A_k^\pi$ denotes a squared matrix of order $k$ retrieved from a permutation of $D^2u(x)$.

Part 1 ($\Rightarrow$) WTS: $u$ is concave $\Rightarrow v_i''\leq 0\ \forall i$. As $u$ is concave, $(-1)^1|A_1^\pi|\geq 0\ \forall\pi$, which means $v_i''\leq 0\ \forall i$.

Part 2 ($\Leftarrow$) WTS: $v_i''\leq 0\ \forall i\Rightarrow u$ is concave. As $v_i''\leq 0\ \forall i$, it is straightforward to verify that $(-1)^k|A_k^\pi|\geq 0\ \forall k,\pi$, which means $u$ is concave. This completes the proof.