Eco 387L (24): Mathematical Economics Fall 2006
Discussion 1

If the **universe** is $M = (0, 10]$ and the metric is Euclidean $d(x, y) = |x - y|$. We want to check if $E = (0, 5]$ is closed or not. We have the same question to set $F = (5, 10]$ and set M itself.

Definition 1 (closedness). A subset $E \subset M$ is closed **in** M if $\forall \{x_k\} : (\forall k\ x_k \in E) \wedge (x_k \to x)$ we must have $x \in E$.

Definition 2 (openness). A subset $E \subset M$ is open **in** M if $\forall x \in E\ \exists r > 0 : B(x, r) \subset E$, where $B(x, r) = \{x' \in M : d(x, x') < r\}$.

(i) WTS: $E = (0, 5]$ is closed. We only need to check the lower end of E because we are worry that $\exists \{x_k\} : x_k \to 0$, e.g. $x_k = 1/(k + 1000)$. However, note that $0 \notin E$ and 0 cannot be the convergent point of any sequence; hence sequences like $x_k = 1/(k + 1000)$ are not convergent (even though they are convergent in $(\mathbb{R}, d)$). By Definition 1, to check for closedness, we only look at convergent sequences $\{x_k\} : x_k \to x > 0$; pick any of those sequence, say $\{x_k\}$, and observe that $\forall k \in \mathbb{N}\ x_k \leq 5$; together, these conditions guarantee that $x \in E$. Thus E is closed. By checking at 5, E cannot be open; Why?

(ii) WTS: $M = (0, 10]$ is both closed and open! The establishment of closedness follows the same line as in (i). Why is M open? M is readily open at the lower end; at 10, by Definition 2, $\exists r > 0$ s.t. the open ball $B(10, r)$ entirely lies in M. Note the requirement "$x' \in M$" in Definition 2.

(iii) WTS: $F = (5, 10]$ is open. Note that $F = E^C$; thus F is open in M. By the same arguments above, you can directly check for openness of F. You may want to compare the lower ends of sets E and F.

Note the difference between completeness and closedness in this context. The metric space (M, d) is closed but not complete because there exists some Cauchy sequence that does not converge in M. Convince yourself with some example.

Finally, bear in mind that openness and closedness depend on the set M and the metric d imposed on that set because those objects determine the convergence criteria and the construction of open balls. In addition, though "small", the quantifiers $\exists$ and $\forall$ play critical roles in definitions and proofs.